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LETTER TO THE EDITOR

Closed solution for the spatially homogeneous Kac's model of the nonlinear Boltzmann equation

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Abstract. We find a closed spatially homogeneous solution of the nonlinear Kac's model in $1+1$ dimensions (velocity v and time t). Choosing for the even velocity part of the distribution function the Bobylev–Krook–Wu mode, we add an odd velocity part. We find the possibility of the existence of the Tjon relaxation effect, when the time t is increasing. This depends on both the initial condition and the cross section.

The well known Bobylev (1975), Krook and Wu (1976) solution (hereafter called BKW even mode) of the Boltzmann equation is a closed, non-trivial solution depending on v^2 and t (velocity v and time t). It is the product of a Maxwellian with time dependant width by a polynomial of the first order in v^2. For the spatially homogeneous Kac's model (Kac 1956, Uhlenbeck and Ford 1963) the BKW even mode was derived by Ernst (1979, 1981). Decomposing the distribution function $f(v, t)$ into its even and odd parts with respect to v, $f=f^+ + f^-$, it corresponds to f^+.

Some years ago Tjon (1979) discovered a very interesting effect. Let us define the reduced distribution function $F(v, t) = f(v, t)/f(v, \infty)$. Then for large but fixed v, t increasing and going to infinity, f may relax to the Maxwellian distribution function (or $F\to 1$) either in a monotonic way $F \leqslant 1$ or with an overpopulation at intermediate time where we have for these fixed v:$F>1$, $F\to 1^+$, $t\to\infty$. Equivalently when the effect exists (depending on initial conditions) it may produce at intermediate times a population of high velocity particles larger than the one present at initial time or at equilibrium.

So depending whether or not this effect exists, we can define two classes of Boltzmann solutions. It was shown (Cornille and Gervois 1980) that the BKW even mode cannot exhibit this effect because necessarily $F \leqslant 1$ for $|v|^2$ higher than some fixed value. Due to this drawback, the BKW even mode cannot represent a general feature for the relaxation to equilibrium of the Boltzmann solutions.

Our aim here is to introduce a non-trivial odd velocity part $f^-(v, t)$, where $f^+(v, t)$ is reduced to the BKW even mode, in such a way that the complete solution $f^+ + f^-$ is a closed one. As a supplement we investigate whether or not the Tjon effect can exist for the complete solution. The Kac's model depends on the three variables v, t, and

x the position:

$$(\partial_t + v\partial_x) f(v) = \int_{-\pi}^{+\pi} \sigma(\theta) \int_{-\infty}^{+\infty} (f^+(v')f^+(w') - f^+(v)f^+(w))\, \mathrm{d}w\, \mathrm{d}\theta$$
$$v' = v\cos\theta - w\sin\theta \qquad w' = v\sin\theta + w\cos\theta \qquad \sigma(\theta) = \sigma(-\theta) \tag{1}$$
$$\int_{-\pi}^{+\pi} \sigma(\theta)\, \mathrm{d}\theta = \sigma_0 = 1$$

where $f(v)$ means $f(v, x, t)$ and $\sigma(\theta)$ is the cross section. We restrict ourselves to the spatially homogeneous case $f = f(v, t)$. We will find that the existence of a closed complete $f(v, t)$ solution *requires a condition* for $\sigma(\theta)$ (which was not the case for the BKW even mode) and the existence of the Tjon effects depends not only on the *initial condition* (macroscopic condition) but also on a further *condition on* $\sigma(\bar{\theta})$ (microscopic condition). In order to understand this surprising fact, it may be useful to recall recent results obtained for the stationary spatially inhomogeneous Kac's distribution function $f(v, x)$. Firstly in that case if, as it seems reasonable, we assume $\sigma(\theta) = \sigma(\pi - \theta)$ then (Cornille *et al* 1983) no closed similarity solutions for $f(v, x)$ exist. Secondly if we do not assume $\sigma(\theta) = \sigma(\pi - \theta)$, then closed similarity solutions exist (Cornille 1983) but the moments

$$\tau_m = \int_{-\pi}^{+\pi} (\cos\theta)^m \sigma(\theta)\, \mathrm{d}\theta \qquad \sigma_{2m} = \int_{-\pi}^{+\pi} (\cos\theta\sin\theta)^{2m} \sigma(\theta)\, \mathrm{d}\theta > 0 \qquad \sigma_0 = \tau_0 \tag{2}$$

must satisfy well defined linear relations. When the complexity of the odd part augments then the number of linear relations also increases (Cornille 1984). In the $f(v, t)$ case the assumption $\sigma(\theta) = \sigma(\pi - \theta)$ decouples entirely the equations for f^+ and f^- leading for the odd part (Ernst 1981) to the trivial solution $f^-(v, t) = e^{-t} f^-(v, 0)$ without any link with the even part f^+. Here we do not retain this special symmetry for $\sigma(\theta)$ and deduce a closed complete solution where f^- (the BKW odd mode) is the partner of the BKW even mode, but the moments σ_2, τ_1, τ_3 of the cross section have to satisfy a linear relation.

In order to find closed solutions we can use different methods: direct substitution of the ansatz solution into the integral equation or expansions with Laguerre polynomials. The Kac's equations for $f(v, t)$ are

$$\partial_t f^+ = \int_{-\pi}^{+\pi} \sigma(\theta) \int_{-\infty}^{+\infty} (f^+(v', t)f^+(w', t) - f^+(v, t)f^+(w, t))\, \mathrm{d}w\, \mathrm{d}\theta \tag{3a}$$
$$\partial_t f^- = \int_{-\pi}^{+\pi} \sigma(\theta) \int_{-\infty}^{+\infty} (f^-(v', t)f^+(w', t) - f^-(v, t)f^+(w, t))\, \mathrm{d}w\, \mathrm{d}\theta. \tag{3b}$$

We seek solutions of the type

$$f^+ = \frac{1}{\sqrt{2\pi}} \exp(-b(t)v^2/2) \sum_0^{n_+} \alpha_{2n}(t) \left(\frac{v}{\sqrt{2}}\right)^{2n}$$
$$f^- = \frac{1}{\sqrt{2\pi}} \exp(-b(t)v^2/2) \sum_0^{n_-} \alpha_{2n+1}(t) \left(\frac{v}{\sqrt{2}}\right)^{2n+1}. \tag{4}$$

Substituting (4) into (3) we find only one possibility $n_+ = 1$, $n_- = 0$. In order that this letter be self-contained we sketch briefly the determination of the BKW even mode (Krook and Wu 1979, Ernst 1981). In equation (3*a*) we find a v^{2m} polynomial such that the coefficients of v^4, v^2, v^0 are zero and furthermore we have the mass and energy conservations laws: $\int f\,dv=\int v^2 f\,dv=1$.

$$-b_t=\frac{\alpha_2\sigma_2}{b^{1/2}} \qquad \alpha_{2t}-\alpha_0 b_t=\frac{-3\alpha_2^2\sigma_2}{b^{3/2}} \qquad \alpha_{0t}=\frac{3}{4}\frac{\alpha_2^2\sigma_2}{b^{5/2}}$$

$$1=\frac{\alpha_0}{b^{1/2}}+\frac{\alpha_2}{2b^{3/2}}=\frac{\alpha_0}{b^{3/2}}+\frac{3}{2}\frac{\alpha_2}{b^{7/2}}. \tag{5a}$$

Integrating (5*a*) we find $\alpha_0=\frac{1}{2}b^{1/2}(3-b)$, $\alpha_2=b^{3/2}(b-1)$, $b=(1-c\mathrm{e}^{-\sigma_2 t})^{-1}$. Substituting the f^+, BKW even mode, into (3*b*) we obtain another polynomial with two terms v and v^3. We find two relations

$$-b_t=\frac{\alpha_2}{b^{1/2}}(\tau_1-\tau_3) \qquad \frac{\alpha_{1t}}{\alpha_1}=-\tau_0+\frac{\alpha_0\tau_1}{b^{1/2}}+\frac{\alpha_2}{2b^{3/2}}(3\tau_3-2\tau_1) \tag{5b}$$

that we integrate taking into account the explicit expressions of α_2, α_0 in terms of b and obtain the BKW odd mode

$$\sigma_2=\tau_1-\tau_3 \qquad \alpha_1(t)=d\,b^{3/2}\exp[-(\tau_0-\tau_1)t] \qquad \tau_0=1. \tag{5b'}$$

Finally the complete BKW solution is

$$f(v,t)=\sqrt{\frac{b}{2\pi}}\exp(-bv^2/2)\left[\frac{3-b}{2}+d\,b\frac{v}{\sqrt{2}}\exp[-(\tau_0-\tau_1)t]+b(b-1)\frac{v^2}{2}\right] \tag{6}$$

where $b=[1-c\,\mathrm{e}^{-\sigma_2 t}]^{-1}$ and $\sigma(\theta)$ must satisfy $\sigma_0=1$, $\sigma(\theta)\neq\sigma(\pi-\theta)$ and the moments relation $\sigma_2-\tau_1+\tau_3=0$.

Firstly, we discuss the restrictions on $\sigma(\theta)$. We remark that if $\sigma(\theta)=\sigma(\pi-\theta)$, then $\tau_1=\tau_3=0$, leading in equation (5*b*) to $\alpha_1=\mathrm{e}^{-t}$ and $b=$ *constant*; a trivial solution which is not linked to the BKW even mode. However it is very easy to construct cross sections satisfying the moments relation and $\sigma(\theta)\neq\sigma(\pi-\theta)$. We give two examples: one with δ distribution functions and another with smooth functions of θ:

$$\sigma(\theta)=\tfrac{1}{4}[(1+\cos\theta_1)(\delta(\theta-\theta_1)+\delta(\theta+\theta_1))+(1-\cos\theta_1)(\delta(\theta-\pi+\theta_1)+\delta(\theta+\pi-\theta_1))] \tag{7a}$$

where $0<|\cos\theta_1|<1$ is arbitrary, and

$$\sigma(\theta)=(57/(16)(17))\cos\tfrac{1}{2}\theta(1+\tfrac{11}{19}\cos\theta). \tag{7b}$$

Secondly we discuss the positivity property of $f(v,t)$. We require that the discriminant of the quadratic form in the bracket of equation (6) is not positive or $(\frac{1}{2}d^2-[(2-3\,c\,\mathrm{e}^{-\sigma_2 t})/(1-c\,\mathrm{e}^{-\sigma_2 t})]\,c\exp[(-\sigma_2+2\tau_0-2\tau_1)t])\leqslant 0$. At $t=0$ this leads to

$$c\in\,]0,\tfrac{2}{3}[\,d\in\,]-\sqrt{2}(\sqrt{3}-1),\sqrt{2}(\sqrt{3}-1)[\qquad d^2\leqslant 2c(2-3c)/(1-c) \tag{8}$$

and this is sufficient at $t>0$ too if we remark $2\tau_0-2\tau_1-\sigma_2=\int\sigma(\theta)(2+\cos\theta)(1-\cos\theta)^2\mathrm{d}\theta>0$.

Thirdly, we discuss the Tjon effect. On the one hand *this effect depends on the initial conditions* (macroscopic condition). In figure 1 we give an example $c=\frac{1}{2}$, σ in (7*b*) with a weak odd part: $d=-0.1$; the relaxation to $F=1$ is from below showing no Tjon effect. In another example, figures 2 and 3, $c=1/2$, σ belongs to (7*a*) with

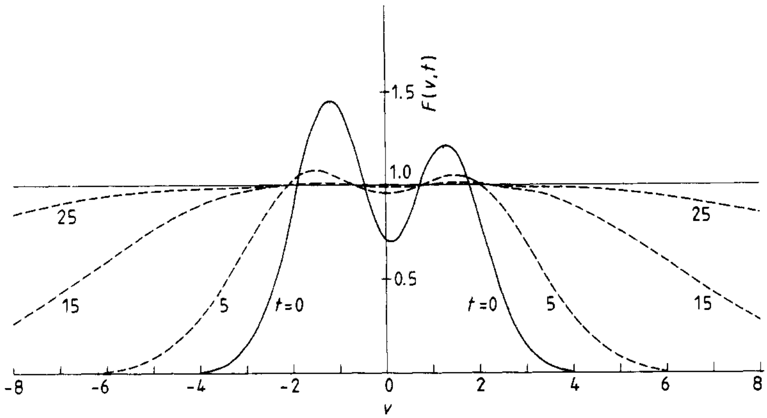

Figure 1. Plot of $F(v, t)$ against v for $d=-0.1$, $c=0.5$ and $\sigma(\theta)$ given by (7*b*).

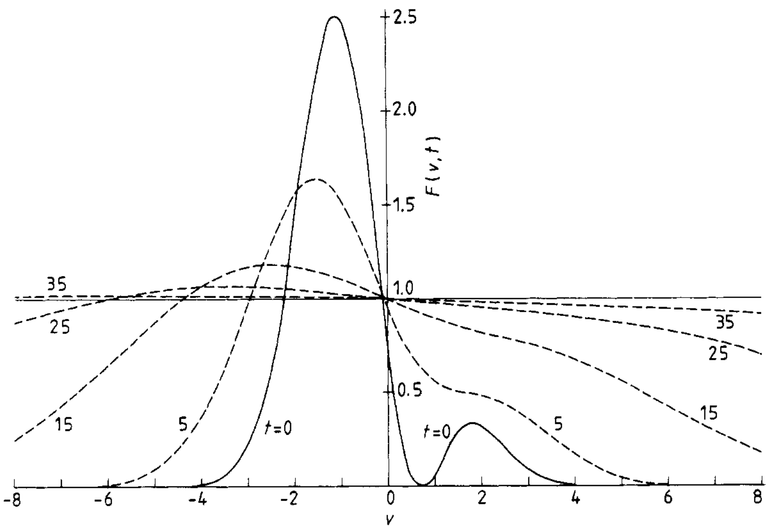

Figure 2. Plot of $F(v, t)$ against v for $d=-1$, $c=0.5$, $\cos\theta_1=0.93$ and $\sigma(\theta)$ given by (7*a*).

$\cos\theta_i=0.93$, and a strong odd part $d=-1$, the effect exists for negative large v values (note if $d=1$ it would be for $v>0$ due to the symmetry $d\to-d$, $v\to-v$). On the other hand *this effect depends on microscopic conditions on* $\sigma(\theta)$. In figure 4 we show the relaxation for the same initial values as in figure 2: $c=\frac{1}{2}$, $d=-1$, σ of the equation (7*a*) type but $\cos\theta_1=0.5$ and we see that no effect exists. Extending our analysis to the case where $\sigma(\theta)$ is given by equation (7*b*) and exploring the whole family of reduced distribution functions $F(v, t)$ for all possible c, d values satisfying the conditions written down in equation (8) we never find the Tjon effect.

A qualitative understanding of the effect can be obtained in the following way. Let us call $v_-(0)$ ($v_+(0)$) the last negative (positive) zero of $F(v, 0)-1$. When t increases two extreme cases can occur: (i) either $v_-(t)(v_+(t))$ does not move or moves slightly towards a finite limiting value, then for $v<v_-(\infty)$ ($v>v_+(\infty)$) $F\leq 1$ and no

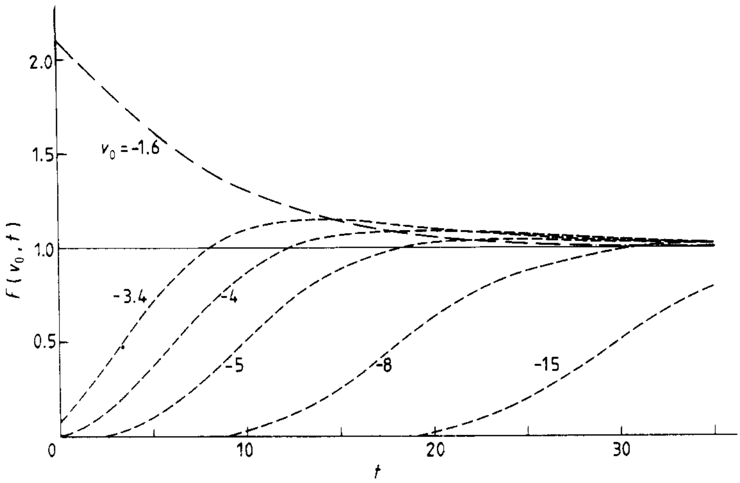

Figure 3. Plot of $F(v, t)$ against t for the same solution as in figure 2.

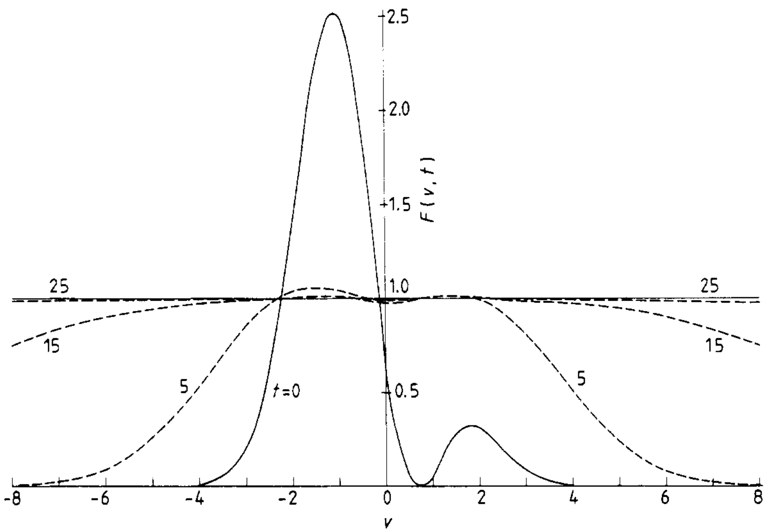

Figure 4. Plot of $F(v, t)$ against v for $d=-1$, $c=0.5$, $\cos\theta_1=0.5$ and $\sigma(\theta)$ given by (7a).

effect appears; or (ii) $v_{\pm}(t)$ or at least one of them goes to infinity and then large v values and corresponding times for which $F \geq 1$ always exist and we have the effect. Of course the transition regime then occurs between two cases. In order to understand the different regimes which appear in figures 2 and 4 it is clearly necessary to interpolate other possible modes of relaxation between these two cases. We have calculated a set of different $F(v, t)$ for the same initial condition $c=\frac{1}{2}$, $d=-1$. $\sigma(\theta)$ belonging to (7a) but $\cos\theta_1$ varying from 0 to 1. The interesting zero is v_-(v_+ if $d=+1$). For $0<\cos\theta_1<0.7$ this zero does not move, near $\cos\theta_1=0.75$ the zero begins to move and at $\cos\theta_1>0.8$ the zero is going to $-\infty$. For $0.8<\cos\theta_1<0.94$ we observe the Tjon effect. However for $\cos\theta_1>0.95$ the zero slows down and the effect disappears.

Practically we say that the *Tjon effect really exists* if both the *zero* $x_-(t)$ (of $F-1$ for $d<0$) *is increasing not too slowly when* $t\to\infty$ and the $F>1$ values are *substantially larger than one.*

One can prove that for the v values for which sign $(vd)<0$, then the effect does not exist. For $vd<0$, F is less than the contribution given by the BKW even mode alone (see equation (6)), and $F\leqslant 1$ for $\frac{1}{2}v^2>4$ (Cornille and Gervois 1980). When the effect exists we do not have the reversed inequality $F\geqslant 1\ \forall v\ \forall t$ sufficiently large, but only for those $|v|$ less than the location of the above discussed zero $|v_\pm(t)|$. Then we can check numerically that the zero is moving in a velocity interval large compared to the one where the distribution is significant at $t=0$. For $c=\frac{1}{2}$, $\cos\theta_1=0.93$ and $d=-1, -0.01$ we find for the zero $v_-(t)$: $v_-(0)\simeq -2, -2$, $v_-(25)\simeq -6, -2.5$, $v_-(50)\simeq -14, -3.5$, $v_-(100)\simeq -75, -16$.

Another method, for the study of the solutions of equation (1), is the use of the Laguerre expansion. It turns out (Cornille 1983, 1984) that the Laguerre polynomials are $L_n^{-(1/2)}$, $L_n^{+(1/2)}$ for f^+, $v^{-1}f^-$

$$f^+\sqrt{2}\pi e^{v^2/2}=\sum L_n^{-1/2}\left(\frac{v^2}{2}\right)(-1)^n D_n^+(x,t)$$

$$f^-\sqrt{2}\pi e^{v^2/2}=\sum L_n^{1/2}\left(\frac{v^2}{2}\right)(-1)^n D_n^-(x,t) \tag{9}$$

and we formally find for the Laguerre moments:

$$\partial_t D_n^+ + \partial_x((n+\tfrac{1}{2})D_n^- + nD_{n-1}^-)=\sum_0^n D_q^+D_{n-q}^+B_{qn}C_n^q$$

$$B_{0n}=\tau_{2n}-\tau_0 \qquad B_{qn}=\sum_0^q(-1)^m C_q^m\tau_{2(n+m-q)} \tag{10a}$$

$$\partial_t D_n^- + \partial_x(D_{n+1}^+ + D_n^+)=\sum_0^n D_q^+D_{n-q}^-E_{qn}C_n^q$$

$$E_{0n}=\tau_{2n+1}-\tau_0 \qquad E_{qn}=\sum_0^q(-1)^m C_q^m\tau_{2(n+m-q)+1} \tag{10b}$$

in equation (10a), the RHS for $n=0, 1$ are zero and correspond to conserved quantities. Now we restrict to the spatially homogeneous $D_n^\pm(t)$ case. As the reader can easily check the BKW complete closed solution equation (6) is

$$D_n^+(t)=(-1)^n(1-n)\omega^n \qquad D_n^-=(-1)^n d\omega^n\exp[-(\tau_0-\tau_1)t]$$
$$\omega=c\,e^{-\sigma_2 t} \qquad \sigma_2-\tau_1+\tau_3=0 \qquad \tau_0=1. \tag{6$'$}$$

We can now discuss the Tjon effect in a semi-theoretical way. We use arguments, when an odd velocity part is present, similar to those of Hauge and Praestgaard (1981) for even velocity distribution alone. Let us define $\bar\omega=\frac{1}{2}\omega v^2$ as a scaling variable, substitute the solution equation (6′) into the Laguerre expansion equation (9), retain the first Laguerre odd and even terms and put to zero terms independent of $\bar\omega$ (they vanish when $t\to\infty$). We obtain for the reduced distribution $F(v,t)=F(\bar\omega,t)$ a rough estimate

$$F_{-1}\simeq \operatorname{sign}(v)dc^{-1/2}\bar\omega^{1/2}e^{-\alpha t}+\tfrac{3}{2}c\bar\omega\,e^{-\sigma_2 t}-\tfrac{1}{2}\bar\omega^2 \tag{11}$$

where $\alpha=\tau_0-\tau_1-\sigma_2/2=\frac{1}{2}\int\sigma(\theta)(1-\cos\theta)(2-\cos\theta-\cos^2\theta)\,d\theta$ is positive. If $d=0$, the odd contribution disappears and we recover the result for the BKW even mode. In order that the odd terms reverse the situation we see that we must require two conditions.

(i) d sign $v>0$ and at fixed c, $|d|$ be as large as possible, which means to include the largest possible odd component. This explains quite well the difference between figure 2, $d=-1$, $c=0.5$, $\cos\theta_1=0.93$ ($d=-1$ is the largest possible value) and other cases with the same c, $\cos\theta_1$, and $d=-0.01, -0.1$ where the zero $x_-(t)\to\infty$ very slowly and the $F>1$ values are very close to 1.

(ii) α small and let us consider for $\sigma(\theta)$ the family (7a): then $\alpha=\frac{1}{2}(1-\cos^2\theta_1)(2-\cos^2\theta_1)$, α decreases when $|\cos\theta_1|$ increases and we recall that this is in accord with the phenomenological description given above. Further we want to explain the numerical result that exists for $|\cos\theta_1|$ a critical value for the displacement of the $F-1$ zero. If we look at a zero of $F-1$ in equation (11) we find that $(|d|/c^{1/2})-(|v|^3/4\sqrt{2})\,e^{(\alpha-3\sigma_2/2)t}(1-6c/v^2)$ must vanish and if we require that the location of the zero increases when $|v|$ increases we obtain $\alpha<\frac{3}{2}\sigma_2$ or $\tau_0-\tau_1-2\sigma_2<0$ or $\int\sigma(\theta)(1-\cos\theta)(1-2\cos^2\theta(1+\cos\theta))d\theta<0$. Applying this result to the $\sigma(\theta)$ family equation (7a) where $\alpha-\frac{3}{2}\sigma_2=(1-\cos^2\theta_1)(1-2\cos^2\theta_1)$ we find that the zero can move for $|\cos\theta_1|$ larger than the critical value $|\cos\theta_1|=(\sqrt{2})^{-1}\simeq0.707$ which is a good result if we recall the crude approximation leading to equation (11). On the contrary for the case of figure 1 (σ belonging to (7b)) we have $\alpha-\frac{3}{2}\sigma_2\simeq0.24>0$, in accord with the non-existence of the Tjon effect.

In conclusion conditions on $\sigma(\theta)$ and $f^-(v,0)$ control respectively the displacement of the zero $x_-(t)$ (if $d<0$) and the possibility of substantial $F>1$ values. The best Tjon effect is obtained taking the most favourable conditions on both $\sigma(\theta)$ and $f^-(v,0)$.

Finally one can ask the following question: is the BKW odd mode given here in equations (6)–(6′) the only non-trivial one that we can add to the BKW even mode? Let us give up the requirement of writing a closed solution. In equation (10b) we substitute the $D_n^+(y)$ given by equation (6′) and integrate. Then the $D_n^-(t)$ are recursively obtained once we know the set $D_n^-(0)$. We must first require that $f(v,0)$ (or $D_n^{\pm}(0)$) is positive and second that the Laguerre series constructed with the $D_n^-(t)$ exist for all t values. We must also have $\sigma(\theta)\neq\sigma(\pi-\theta)$ but may be no other very important conditions on $\sigma(\theta)$. We are investigating this problem.

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